https://doi.org/10.33805/2641-7383.112
Volume 2 Issue 1 | PDF 112 | Pages 13

EDELWEISS PUBLICATIONS
OPEN ACCESS

Edelweiss Chemical Science Journal

Research Article ISSN: 2641-7383

Matter Dynamics in a Unitary Relativistic Quantum Theory

Eliade Stefanescu*

Affiliation: Advanced Studies in Physics Centre of the Romanian Academy, Academy of Romanian Scientists, Bucharest, Romania

***Corresponding author:** Eliade Stefanescu, Professor, Advanced Studies in Physics Centre of the Romanian Academy, Academy of Romanian Scientists, 010702 Bucharest, Romania, Tel: +4021-3188106, E-mail: eliadestefanescu@yahoo.fr

Citation: Stefanescu E. Matter dynamics in a unitary relativistic quantum theory (2019) Edelweiss Chem Sci J 2: 27-39.

Received: Nov 11, 2019
Accepted: Nov 20, 2019
Published: Nov 26, 2019

Copyright: © 2019 Stefanescu E. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

Abstract

We describe the matter dynamics as a positively defined density $\rho\left(x^i,t\right)=M\left|\psi\left(x^i,t\right)\right|^2$ and show that, according to the general theory of relativity, such a distribution can be conceive only as of a fragment of matter with a finite mass M equal to a mass M_0, as a characteristic of the matter dynamics, $M=M_0$ — the matter quantization. The group velocities of the Fourier conjugate representations in the coordinate and momentum spaces describe the dynamics of a quantum particle in agreement with the Hamiltonian equations. Under the action of an external (non-gravitational) field, the acceleration of the quantum matter has two components: **1)** A component perpendicular to the velocity, given by the relativistic mechanical component of the time dependent phase, and **2)** A component given by the additional field terms of this phase. A free quantum particle is described by a non-dispersing wave function $\psi\left(x^i,t\right)$, contrary to the solution of the Schrödinger equation. A coherent electromagnetic field, in resonance with a system of active quantum particles in a Fabry-Perot cavity, has a wave vector approximately proportional to the metric elements, as the resonance frequency is approximately constant – a gravitational wave can be detected by the transmission characteristics of an active Fabry-Perot cavity. In a constant gravitational field, a quantum particle undertakes a velocity and an acceleration, which, at the boundary of a black hole are null – absorption and evaporation processes at the boundary of a black hole arise only by gravitational perturbations. Generally, a quantum particle is described by a time-space volume, called graviton, with a spin 2, and a distribution of a specific matter in this volume, with a half-integer spin for Fermions and an integer spin for Bosons. A graviton Lagrangian is obtained as a curvature integral on a graviton volume, and a Hamiltonian tensor is obtained for the gravitational coordinates and velocities.

Keywords: Quantum particle, Wave packet, Group velocity, Lagrangian, Hamiltonian, Maxwell equations, Spin, Metric tensor, Covariant derivation, Gravitational wave, Graviton spin.

Introduction

In spite of the protests and the arguments of some very important physicists as Louis de Broglie, Erwin Schrödinger, and Albert Einstein, the triad Niels Bohr, Werner Heisenberg, and Wolfgang Pauli, succeeded to impose the presently official quantum mechanics, based on the particle-wave dualism and the uncertainty principle, where only an abstract wave function is described, as the matter itself is considered of the form of dots with coordinates determined only with a probability given by this wave function, as a solution of the Schrödinger equation [1-5]. However, during the long history of about of a century of this theory, it continued to seem strange and questionable – famous physicists found this theory non-understandable, and various philosophical approaches have been tried as logical grounds of its validity [6-11]. More than that, very recently, the fundamentals of physics became an active subject of debate [12-19].

We found that the Schrödinger equation, standing at the basis of this theory is deficient, being in disagreement with the Hamilton equations and, consequently, with the basic principle of the energy conservation – an equation in agreement with these fundamentals is obtained only when the Hamiltonian of the Schrödinger equation is replaced with the Lagrangian [20-22]. With the relativistic Lagrangian, the relativistic principle of the invariance of any time-space interval takes the form of a relativistic quantum principle of invariance of the time dependent phase of any quantum particle. At the same time, we remark that the Lagrangian is no more a simple characteristic, which for certain energy could be attached to a simple dot, but a time dependent function describing the matter dynamics.

In section 2, we come back to the fundamentals. The matter dynamics is described as a time-space dependent density, positively defined by the square of the distribution function $\psi\left(x^i,t\right)$ with the total mass M as a coefficient,

$$\rho\left(x^i,t\right)=M\left|\psi\left(x^i,t\right)\right|^2, \qquad (1)$$

according to the general theory of relativity [23]. From the invariance of the time-space interval, which leads to the Lorentz transformation, we obtain that any acceleration induced by an external (non-gravitational) field is perpendicular to the velocity. In this case, a distribution function can be considered as a Fourier integral of plane waves perpendicular to the velocity.

When the coordinate coefficient of the phase of such a wave is considered as a derivative with the velocity of the time coefficient of this phase, the group velocity of the reverse Fourier transform takes the form of the Lagrange equation. This means that the time coefficient of the wave packet $\psi\left(x^i,t\right)$ of a matter distribution (1) is the Lagrangian of a particle with a mass M_0 as a characteristic of the density dynamics. This means that the mass M, initially consider in (1) as an arbitrary coefficient, must be equal to the mass M_0 as a characteristic of the matter dynamics. This means that the matter dynamics characterized by a specific mass M_0, can be understood only for fragments of a mass $M=M_0$, called quantum particles. It is remarkable that in the two wave functions, describing the matter dynamics in the coordinate and momentum spaces according to the general theory of relativity, the phase velocities are equal to the group velocities, which means that these are wave functions, unlike the wave functions provided by the conventional Schrödinger equation, are not dispersing during the time evolution. In section 3, we obtain a relativistic invariant of the matter density, and a relativistic equation of the matter conservation.

In section 4, we consider a quantum particle in an electromagnetic field. In this way, the particle rapidly varying waves with time dependent phases depending on the metric tensor are modulated by additional terms of these phases proportional to the electromagnetic potentials – a unitary electromagnetic gravitational theory of the quantum dynamics is obtained. We consider an active Fabry-Perot cavity in a gravitational wave, and obtain a wave equation for the wave vector of an active quantum particle in the cavity. In this way, for a quasi-resonant electromagnetic wave in this cavity, we obtain an oscillating cavity detuning, leading to an oscillation of the transmission characteristic of the cavity.

In section 5, we consider the gravitational field according to Einstein's law of the time-space curvature. From the geodesic equations, we obtain the acceleration of a quantum particle in a gravitational field, as a generalization of the Newtonian equation, where, for a black hole, falls to null at its boundary. In a constant gravitational field, no particle enters the black hole, and no particle gets out from the black hole. By integrating the geodesic equations in a central field, we obtain the radial velocity which, for a black hole, takes a null value: no particle passes the boundary of a black hole. However, in the realistic cases, the gravitational field is not constant, undertaking fluctuations induced by the matter moving around, inducing a penetrability of the black hole boundary. By these fluctuations, absorption and evaporation processes of a black hole arise.

In section 6, we consider a quantum particle in a gravitational wave. From the group velocity in the momentum space, which is of the form of the Lagrange equation, and the geodesic equations, we obtain dynamic equations as functions of the metric tensor. In the realistic case of a weak interaction with a gravitational wave, we consider a metric tensor of the first order in coordinates, with a polarization vector for the coordinate dynamics, and an amplitude tensor. We obtain an acceleration of the quantum particle equal to the total amplitude of the gravitational wave. More than that, we obtain an invariant of the of the matric tensor, with two terms transforming one another by rotation with an angle $\pi/2$. From the rotational operator as a function of the angular momentum operator, we obtain the eigenvalue 2, we call graviton spin. For a metric tensor of the second order in coordinates, we obtain a harmonic oscillation of a quantum particle in a gravitational wave. In section 7, we define a graviton Hamiltonian tensor. We show that, for a first order metric tensor in a gravitational wave, the diagonal elements of the energy cancel one another, as the total curvature energy is null. Section 8 is for conclusions.

The Quantization Condition of the Matter Dynamics

We consider the physical universe as a distribution of intrinsic matter described by space distances and time intervals as it has physical characteristics. It is interesting that this distribution of matter in a system S of time-space coordinates,

$$x=\left(x^{\alpha}\right)=\left(x^0=ct,x^1,x^2,x^3\right)=\left(x^0=ct,x^i\right), \qquad (2)$$

is curved as a hyper surface in a system of a larger number of universal coordinates [23],

$$y^n=y^n\left(x^{\nu}\right), \qquad n=1,2,\ldots N, \qquad \nu=0,1,2,3, \qquad (3)$$

under the action of a gravitational field generated by an extrinsic matter described by a density of the form (1). In this case, the differential time-space interval ds takes the form

$$\mathrm{ds}^2=h_{nm}\mathrm{d}y^m\mathrm{d}y^n=h_{nm}y^m{}_{,\mu}y^n{}_{,\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}=g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}, \quad h_{nm}=h_{nm}, \qquad (4)$$

depending on the metric tensor

$$g_{\mu\nu}=g_{\nu\mu}=h_{nm}y^m{}_{,\mu}y^n{}_{,\nu}=y^m{}_{,\mu}y_{m,\nu}. \qquad (5)$$

In the absence of any gravitational field, when the space is flat, the differential time-space interval takes a simpler form

$$\mathrm{ds}^2=\mathrm{d}x^{0^2}-\mathrm{d}x^{1^2}-\mathrm{d}x^{2^2}-\mathrm{d}x^{3^2}, \qquad (6)$$

describing a time evolution with a constant velocity V, which in a system with the axis X^1 in this direction, is

$$\frac{V}{c}=\frac{\mathrm{d}x^1}{\mathrm{d}x^0}=\frac{\mathrm{d}x^1}{c\mathrm{d}t}. \qquad (7)$$

For a reference system S' with parallel axes, propagating with this velocity,

$$\begin{aligned} x^0&=x^0\left(x^{0'},x^{1'}\right)\\ x^1&=x^1\left(x^{0'},x^{1'}\right), \qquad x^2=x^{2'}, \qquad x^3=x^{3'}, \end{aligned} \qquad (8)$$

from the invariance of the time-space interval,

$$\mathrm{ds}^2=\mathrm{ds}'^2=\mathrm{d}x^{0'^2}-\mathrm{d}x^{1'^2}-\mathrm{d}x^{2'^2}-\mathrm{d}x^{3'^2}, \qquad (9)$$

we obtain the transformation relations

$$\begin{aligned} \mathrm{d}x^0&=\sqrt{1+x^1{}_{,0'}{}^2}\,\mathrm{d}x^{0'}+x^1{}_{,0'}\mathrm{d}x^{1'}\\ \mathrm{d}x^1&=x^1{}_{,0'}\mathrm{d}x^{0'}+\sqrt{1+x^1{}_{,0'}{}^2}\,\mathrm{d}x^{1'}, \end{aligned} \qquad (10)$$

which takes the form of the Lorentz transformation,

$$\mathrm{d}t=\frac{\mathrm{d}t'+\frac{V}{c^2}\mathrm{d}x^{1'}}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (11)$$
$$\mathrm{d}x^1=\frac{\mathrm{d}x^{1'}+V\mathrm{d}t'}{\sqrt{1-\frac{V^2}{c^2}}},\qquad \mathrm{d}x^2=\mathrm{d}x^{2'},\qquad \mathrm{d}x^3=\mathrm{d}x^{3'}.$$

For a matter element in a reference system S, with a proper reference system S', $\mathrm{d}x^{1'}=\mathrm{d}x^{2'}=\mathrm{d}x^{3'}=0$ in the proper time interval $\mathrm{d}x^{0'}=c\mathrm{d}t'=\mathrm{d}s$, from equation (4) we obtain the fundamental equation

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=1 \qquad (12)$$

for the velocities

$$\dot{x}^\mu=\frac{\mathrm{d}x^\mu}{\mathrm{d}s} \qquad (13)$$

as derivatives coordinate with the proper time of this matter element. From the covariant derivative of the fundamental equation (12) with the property of the null covariant derivative of the metric tensor,

$$g_{\mu\nu;\sigma}=0, \qquad (14)$$

we obtain the relation

$$\dot{x}^\mu_{;\sigma}\dot{x}^\sigma\dot{x}_\mu=0 \qquad (15)$$

of a covariant acceleration $\dot{x}^\mu_{;\sigma}\dot{x}^\sigma$ perpendicular to the velocity, or the simpler relation of a covariant derivative perpendicular to the velocity.

$$\dot{x}^\mu_{;\sigma}\dot{x}_\mu=0. \qquad (16)$$

More than that, with the first relation (11), for a matter distribution with a velocity V,

$$\dot{x}^\mu=\frac{\mathrm{d}x^\mu}{\mathrm{d}s}=\frac{\mathrm{d}x^\mu}{c\mathrm{d}t'}=\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\frac{\mathrm{d}x^\mu}{c\mathrm{d}t}\quad\rightarrow\quad\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\dot{x}^\mu, \qquad (17)$$

with change of notation for the velocities in the reference system S,

$$\dot{x}^\mu=\frac{\mathrm{d}x^\mu}{c\mathrm{d}t}=\frac{V}{c}, \qquad (18)$$

we obtain a relation similar to (12).

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=1-\frac{V^2}{c^2}, \qquad (19)$$

which leads to the same perpendicularity relations (15) and (16). We consider the acceleration of such a matter element under the action of an external (non-gravitational) field A^μ, as a modification of the geodesic equation, depending on an arbitrary parameter, which could be the proper time s, or the time ct in an arbitrary system in motion with a velocity V [23].

$$\dot{x}^\mu_{,\nu}\dot{x}^\nu=-\Gamma^\mu_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma+A^\mu. \qquad (20)$$

Thus, we obtain that any external acceleration is a covariant one,

$$A^\mu=\left(\dot{x}^\mu_{,\nu}+\Gamma^\mu_{\nu\sigma}\dot{x}^\sigma\right)\dot{x}^\nu=\dot{x}^\mu_{;\nu}\dot{x}^\nu. \qquad (21)$$

More than that, by multiplying this acceleration with the velocity, with (16) we obtain that this acceleration is perpendicular to the velocity,

$$A^\mu\dot{x}_\mu=\dot{x}^\mu_{;\nu}\dot{x}^\nu\dot{x}_\mu=0. \qquad (22)$$

In this way, we find that any matter element of a matter distribution (1) in an external field is accelerated by this field, which is perpendicularly to the velocity. This means that a distribution of matter (1) can be described by a Fourier series expansion

$$\psi_0\left(x^i,t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi\left(p_j,t\right)e^{\frac{\mathrm{i}}{\hbar}\left[p_jx^j-L_0\left(x^\alpha,\dot{x}^\alpha\right)t\right]}\mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3, \qquad (23)$$

with a reverse Fourier transform

$$\varphi_0\left(p_j,t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi\left(x^i,t\right)e^{-\frac{\mathrm{i}}{\hbar}\left[p_jx^j-L_0\left(x^\alpha,\dot{x}^\alpha\right)t\right]}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3. \qquad (24)$$

It is interesting that if we consider the wave vector of these waves in the form

$$p_j=\frac{\partial L_0}{c\partial\dot{x}^j}, \qquad (25)$$

the group velocity of the wave packet (24) takes the form of the Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}p_j\equiv c\dot{p}_j\equiv\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_0}{c\partial\dot{x}^j}=\frac{\partial}{\partial x^j}L_0\left(x^\alpha,\dot{x}^\alpha\right). \qquad (26)$$

This means that the coefficient of the time dependent phase is in the form of a Lagrangian

$$L_0\left(x^\alpha,\dot{x}^\alpha\right)=-M_0c^2\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}=-M_0c^2\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j} \qquad (27)$$

of a particle with the mass M_0, where, for simplicity, we considered a time-space diagonalization. From this Lagrangian with the fundamental equation (12) for the proper system, with an explicit expression for a certain coordinate j, we obtain the momentum.

$$p_j=\frac{\partial L_0}{c\partial\dot{x}^j}=-M_0c\frac{\partial}{\partial\dot{x}^j}\sqrt{g_{00}+g_{ik}\dot{x}^i\dot{x}^k}=-M_0c\frac{\partial}{\partial\dot{x}^j}\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j+g_{ji}\dot{x}^j\dot{x}^i}$$
$$=-\frac{M_0c\left(g_{ij}\dot{x}^i+g_{ji}\dot{x}^i\right)}{2\sqrt{g_{00}+g_{ik}\dot{x}^i\dot{x}^k}}=\frac{-M_0cg_{ij}\dot{x}^i}{\sqrt{g_{00}+g_{ik}\dot{x}^i\dot{x}^k}}=-M_0cg_{ij}\dot{x}^i. \qquad (28)$$

With this momentum, the group velocity of the wave packet (23) is

$$\frac{\mathrm{d}}{\mathrm{d}t}x^j = c\dot{x}^j = \frac{\partial L_0}{\partial p^j} = \frac{\partial\left(-M_0c^2\sqrt{g_{00}\dot{x}^{0}\dot{x}^{0}+g_{ik}\dot{x}^i\dot{x}^k}\right)}{\partial\left(-M_0cg_{ij}\dot{x}^j\right)} = c\frac{\partial\left(\sqrt{g_{00}\dot{x}^{0}\dot{x}^{0}+g_{ij}\dot{x}^i\dot{x}^j+g_{ji}\dot{x}^j\dot{x}^i}\right)}{\partial\left(g_{ij}\dot{x}^i\right)}$$

$$= c\frac{\dot{x}^j}{\sqrt{g_{00}\dot{x}^{0}\dot{x}^{0}+g_{ij}\dot{x}^i\dot{x}^j}} = c\dot{x}^j, \tag{29}$$

which means that the wave vector $\frac{1}{\hbar}p_j$ and the frequency $\frac{1}{\hbar}L_0$ given by the expressions (28) and (27), equations (23) and (24) describe waves of matter with every matter differential element moving with the group velocity — equations (23) and (24) describe a non-dispersing dynamics of the distribution of matter.

For the Hamiltonian:

$$H_0 = cp_j\dot{x}^j - L_0\left(x^j,\dot{x}^j,t\right), \tag{30}$$

with the definition (25) of the momentum and the group velocity (26), we obtain an identity.

$$\begin{aligned}\mathrm{d}H_0 &\equiv \frac{\partial H_0}{\partial p^j}\mathrm{d}p_j + \frac{\partial H_0}{\partial x^j}\mathrm{d}x_j + \frac{\partial H_0}{\partial t}\mathrm{d}t \\ &= c\dot{x}^j\mathrm{d}p_j + cp_j\mathrm{d}\dot{x}^j - \frac{\partial L}{\partial x^j}\mathrm{d}x^j - \frac{\partial L}{\partial \dot{x}^j}\mathrm{d}\dot{x}^j - \frac{\partial L}{\partial t}\mathrm{d}t \\ &= c\dot{x}^j\mathrm{d}p_j + \underline{cp_j\mathrm{d}\dot{x}^j} - c\dot{p}_j\mathrm{d}x^j - \underline{cp_j\mathrm{d}\dot{x}^j} - \frac{\partial L}{\partial t}\mathrm{d}t,\end{aligned} \tag{31}$$

This leads to the Hamilton equations

$$\dot{x}^j = \frac{1}{c}\frac{\partial H_0}{\partial p_j}, \qquad \dot{p}_j = -\frac{1}{c}\frac{\partial H_0}{\partial x^j}, \qquad \frac{\partial H_0}{\partial t} = -\frac{\partial L_0}{\partial t}. \tag{32}$$

These equations describe the dynamics of a particle with the mass M_0 as characteristic of the matter dynamics described by the coefficient (27) of the time-dependent phase. Thus, we come to the conclusion that the total mass M of the matter distribution (1) equal to the mass M_0 describing the matter dynamics,

$$\int\rho\left(x^i,t\right)\mathrm{d}^3x = M\int\left|\psi\left(x^i,t\right)\right|^2\mathrm{d}^3x = M = M_0, \tag{33}$$

corresponds to a quantum particle, described by the wave functions (23) and (24) with the normalization conditions

$$\int\left|\psi\left(x^i,t\right)\right|^2\mathrm{d}^3x = 1, \qquad \int\left|\varphi\left(p_i,t\right)\right|^2\mathrm{d}^3p = 1. \tag{34}$$

With the nonrelativistic Lagrangian

$$L_0\left(\vec{r},\dot{\vec{r}}\right) = T\left(\vec{p}\right) - U\left(\vec{r}\right) \tag{35}$$

as a function of the kinetic energy $T\left(\vec{p}\right)$, and the potential energy $U\left(\vec{r}\right)$, the wave functions (23) and (24) take the form

$$\psi_0\left(\vec{r},t\right) = \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi_0\left(\vec{p},t\right)e^{\frac{\mathrm{i}}{\hbar}\{\vec{p}\vec{r}-[T(\vec{p})-U(\vec{r})]t\}}\mathrm{d}^3\vec{p} \tag{36}$$

$$\varphi_0\left(\vec{p},t\right) = \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi_0\left(\vec{r},t\right)e^{-\frac{\mathrm{i}}{\hbar}\{\vec{p}\vec{r}-[T(\vec{p})-U(\vec{r})]t\}}\mathrm{d}^3\vec{r}.$$

For the Hamiltonian

$$H_0\left(\vec{p},\vec{r}\right) = T\left(\vec{p}\right) + U\left(\vec{r}\right), \tag{37}$$

the group velocities of these equations lead to the Hamilton equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{r} &= \frac{\partial}{\partial\vec{p}}T\left(\vec{p}\right) = \frac{\partial H_0}{\partial\vec{p}} \\ \frac{\mathrm{d}}{\mathrm{d}t}\vec{p} &= -\frac{\partial}{\partial\vec{r}}U\left(\vec{r}\right) = -\frac{\partial H_0}{\partial\vec{r}},\end{aligned} \tag{38}$$

which are in agreement with the conservation of the energy $H_0\left(\vec{p},\vec{r}\right) = E$,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial H_0}{\partial\vec{r}}\frac{\mathrm{d}}{\mathrm{d}t}\vec{r} + \frac{\partial H_0}{\partial\vec{p}}\frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = 0. \tag{39}$$

The Conservation of the Quantum Matter

In the previous sections, we showed that a quantum particle is a fragment of a continuous distribution of matter satisfying the quantization condition of a mass given by a dynamic characteristic of this matter. We consider the quantization condition (33) in two reference systems,

$$\int\rho\left(x^{\mu'}\right)\mathrm{d}x^{0'}\mathrm{d}x^{1'}\mathrm{d}x^{2'}\mathrm{d}x^{3'} = \int\rho\left(x^{\mu}\right)\,J\,\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = M_0, \tag{40}$$

which depends on the Jacobian

$$J = \mathrm{Det}\left(x^{\mu'}_{,\alpha}\right). \tag{41}$$

We notice that the elements of the Jacobian are elements of a tensor transformation, which for the metric tensor takes the form

$$g_{\alpha\beta} = x^{\mu'}_{,\alpha}x^{\nu'}_{,\beta}g_{\mu'\nu'}. \tag{42}$$

We obtain a relation between determinants,

$$g = J^2g', \tag{43}$$

which leads to the Jacobian expression

$$J = \frac{\sqrt{-g}}{\sqrt{-g'}}, \tag{44}$$

as a function of the metric tensor determinant

$$g = \mathrm{Det}\left(g_{\alpha\beta}\right), \tag{45}$$

in the two reference systems. With the differential elements

$$\mathrm{d}x^{0'}\mathrm{d}x^{1'}\mathrm{d}x^{2'}\mathrm{d}x^{3'} = J\,\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \tag{46}$$

and the expression (44) of the Jacobian, the quantization relation (40) leads to the integral

$$\int \rho\left(x^{\mu'}\right)\sqrt{-g'}\,\mathrm{d}x^{0'}\mathrm{d}x^{1'}\mathrm{d}x^{2'}\mathrm{d}x^{3'} = \int \rho\left(x^{\mu}\right)\sqrt{-g}\,\mathrm{d}x^{0}\mathrm{d}x^{1}\mathrm{d}x^{2}\mathrm{d}x^{3}, \tag{47}$$

of a density invariant

$$I_D = \rho\left(x^{\mu'}\right)\sqrt{-g'} = \rho\left(x^{\mu}\right)\sqrt{-g}, \tag{48}$$

proportional to the scalar density $\sqrt{-g}$. For the matter flow four-vector

$$J^{\mu} = \rho\dot{x}^{\mu}, \tag{49}$$

we consider the conservation relation of a null covariant divergence,

$$J^{\mu}{}_{:\mu} = J^{\mu}_{,\mu} + \Gamma^{\mu}_{\nu\mu}J^{\nu} = J^{\nu}_{,\nu} + \Gamma^{\mu}_{\nu\mu}J^{\nu} = 0. \tag{50}$$

From the general expression of the second kind Christoffel symbol as a function of the metric tensor,

$$\Gamma^{\mu}_{\nu\sigma} = g^{\mu\lambda}\Gamma_{\lambda\nu\sigma} = g^{\mu\lambda}\frac{1}{2}\left(g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\sigma\nu,\lambda}\right), \tag{51}$$

with the symmetry condition of this tensor, the contracted form of this symbol in (50) takes a form depending on the determinant of the metric tensor,

$$\Gamma^{\mu}_{\nu\mu} = \frac{1}{2}g^{\mu\lambda}\left(\underline{g_{\lambda\nu,\mu}} + g_{\lambda\mu,\nu} - \underline{g_{\mu\nu,\lambda}}\right) = \frac{1}{2}g^{\mu\lambda}g_{\lambda\mu,\nu}$$
$$= \frac{1}{2}g^{-1}g_{,\nu} = \frac{1}{2}(-g)^{-1}(-g)_{,\nu} = \frac{\left(\sqrt{-g}\right)_{,\nu}}{\sqrt{-g}}, \tag{52}$$

as the conservation relation (50) takes the form of an ordinary divergence,

$$J^{\mu}{}_{:\mu}\sqrt{-g} = \left(J^{\mu}\sqrt{-g}\right)_{,\mu} = 0. \tag{53}$$

By integrating this expression on a three-dimensional volume V

$$\int_V \left(J^{\mu}\sqrt{-g}\right)_{,\mu}\mathrm{d}^3x = 0. \tag{54}$$

and separating the time derivative from the spatial derivatives,

$$\int_V \left(J^{0}\sqrt{-g}\right)_{,0}\mathrm{d}^3x = -\int_V \left(J^{m}\sqrt{-g}\right)_{,m}\mathrm{d}^3x, \tag{55}$$

with the Gauss theorem we obtain the conservation relation,

$$\left(\int_V J^{0}\sqrt{-g}\mathrm{d}^3x\right)_{,0} = -\oint_{\Sigma_V} J^{m}\sqrt{-g}\mathrm{d}^2x_m. \tag{56}$$

In the nonrelativistic case,

$$\dot{x}^m << 1: \quad J^0 = \rho\dot{x}^0 \approx \rho, \quad J^m = \rho\dot{x}^m, \tag{57}$$

this expression takes the ordinary form of the matter conservation,

$$\left(\int_V \rho\mathrm{d}^3\vec{r}\right)_{,0} = -\oint_{\Sigma_V}\vec{J}\mathrm{d}^2\vec{r}. \tag{58}$$

Charged Quantum Particle in Electromagnetic and Gravitational Fields

We notice that the wave functions (23)-(24) describe the dynamics of a quantum particle given by the rapidly varying factor with a time dependent phase proportional to mass. However, under the action of an external field, besides the acceleration (21) perpendicular to the velocity (**Figure 1**), the quantum particle gets an additional acceleration due to the additional terms of this field in the time dependent phase.

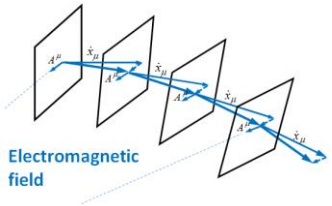

Figure 1: Quantum particle in electromagnetic field evolving in plane waves, accelerated by this field perpendicularly to the velocity.

For a quantum particle in electromagnetic field, the Lagrangian (27) takes a form with additional terms proportional to the vector potential $A_j\left(x^{\mu}\right)$ conjugated to coordinates, in agreement with the Aharonov-Bohm effect [24], and the electric potential $U(x^j)$ conjugated to time,

$$L\left(x^{\alpha},\dot{x}^{\alpha}\right) = -M_0c^2\sqrt{g_{00} + g_{ij}\dot{x}^i\dot{x}^j} - eU\left(x^i\right) + ecA_j\left(x^{\mu}\right)\dot{x}^j, \tag{59}$$

which in the proper time $cdt = \mathrm{d}s$, with the fundamental equation (12) takes the simpler form

$$L\left(x^{\alpha},\dot{x}^{\alpha}\right) \simeq -M_0c^2 - eU\left(x^i\right) + ecA_j\left(x^{\mu}\right)\dot{x}^j. \tag{60}$$

According to the most cases of practical interest, we considered a constant electric potential, as of an atomic nucleus or a quantum dot, and a time dependent vector potential as of an electromagnetic radiation. Obviously under the action of an attraction central field, the particle can finally reach a rotational state round the attraction center (**Figure 2**).

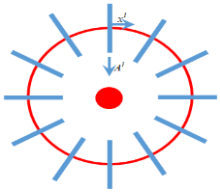

Figure 2: Quantum particle in a central field.

From the Lagrangian (59), we obtain the canonic the momentum

$$P_j = \frac{\partial L}{c\partial\dot{x}^j} = \frac{-M_0cg_{ij}\dot{x}^i}{\sqrt{g_{00} + g_{ik}\dot{x}^i\dot{x}^k}} + eA_j\left(x^{\mu}\right), \tag{61}$$

which in the proper time is

$$P_j = -M_0cg_{ij}\dot{x}^i + eA_j\left(x^{\mu}\right). \tag{62}$$

From the Lagrangian (59) and the momentum (61) we obtain the Hamiltonian of the particle in electromagnetic field,

$$\begin{aligned} H &= cP_j\dot{x}^j - L\left(x^\alpha,\dot{x}^\alpha\right) \\ &= c\left[\frac{-M_0cg_{ij}\dot{x}^i}{\sqrt{g_{00}+g_{ik}\dot{x}^i\dot{x}^k}}+\underline{eA_j\left(x^\mu\right)}\right]\dot{x}^j + M_0c^2\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j}+eU\left(x^i\right)-\underline{ecA_j\left(x^\mu\right)}\dot{x}^j \\ &= \frac{M_0c^2g_{00}}{\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j}}+eU\left(x^i\right), \end{aligned} \tag{63}$$

which does not depend on the time dependent vector potential of the electromagnetic field. For the first term of this expression we calculate

$$\begin{aligned} \frac{M_0^2c^2g_{00}^{\ 2}}{g_{00}+g_{ij}\dot{x}^i\dot{x}^j} &= g_{00}\left(\frac{-M_0^2c^2g_{ij}\dot{x}^i\dot{x}^j}{g_{00}+g_{ij}\dot{x}^i\dot{x}^j}+M_0^{\ 2}c^2\right) = g_{00}\left(\frac{M_0c\dot{x}^j\left(P_j-eA_j\right)}{\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j}}+M_0^{\ 2}c^2\right) \\ &= g_{00}\left(\delta_i^j\frac{M_0c\dot{x}^i\left(P_j-eA_j\right)}{\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j}}+M_0^{\ 2}c^2\right) = g_{00}\left(g^{jk}g_{ki}\frac{M_0c\dot{x}^i\left(P_j-eA_j\right)}{\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j}}+M_0^{\ 2}c^2\right) \\ &= g_{00}\left[M_0^2c^2-g^{jk}\left(P_j-eA_j\right)\left(P_k-eA_k\right)\right]. \end{aligned} \tag{64}$$

Thus, the Hamiltonian (63) takes the canonical form

$$H = c\sqrt{g_{00}\left(M_0^2c^2-g^{ij}\left[P_i-ecA_i\left(x^\mu\right)\right]\left[P_j-ecA_j\left(x^\mu\right)\right]\right)}+eU\left(x^i\right), \tag{65}$$

where the electromagnetic field is described by the electromagnetic potentials $U\left(x^i\right)$, $A_j\left(x^\mu\right)$, as the gravitational field is described by the metric tensor elements g_{00}, g^{ij}, which also depend on coordinates. In a central gravitational field, these elements take the form of the Schwarzschild solution of the equation of gravitation,

$$\begin{aligned} &g_{00} = 1-\frac{2m}{r}, \qquad g_{11} = -\left(1-\frac{2m}{r}\right)^{-1}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta \\ &g^{00} = \left(1-\frac{2m}{r}\right)^{-1}, \quad g^{11} = -\left(1-\frac{2m}{r}\right), \quad g^{22} = -r^{-2}, \quad g^{33} = -r^{-2}\sin^{-2}\theta. \end{aligned} \tag{66}$$

At the same time, the Lagrangian (60) in the proper time takes a form depending on the canonical momentum (62),

$$\begin{aligned} L &= -M_0c^2 - eU\left(x^i\right)+ecA_j\left(x^\mu\right)\delta_k^j\dot{x}^k \\ &= -M_0c^2 - eU\left(x^i\right)+ecA_j\left(x^\mu\right)g^{jk}g_{ki}\dot{x}^i \\ &= -M_0c^2 - e\left[U\left(x^i\right)+g^{jk}A_j\left(x^\mu\right)\frac{P_k-eA_k\left(x^\mu\right)}{M_0}\right]. \end{aligned} \tag{67}$$

With this Lagrangian, the wave functions (23) and (24) take the form

$$\begin{aligned} \psi\left(x^i,t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi\left(P_j,t\right)e^{\frac{i}{\hbar}\left[P_jx^j+e\left[U\left(x^i\right)+g^{jk}A_k\left(x^\mu\right)\frac{P_j-eA_j\left(x^\mu\right)}{M_0}\right]t\right]}e^{-\frac{i}{\hbar}M_0c^2t}\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P_3 \\ \varphi\left(P_j,t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi\left(x^i,t\right)e^{-\frac{i}{\hbar}\left[P_jx^j+e\left[U\left(x^i\right)+g^{jk}A_k\left(x^\mu\right)\frac{P_j-eA_j\left(x^\mu\right)}{M_0}\right]t\right]}e^{\frac{i}{\hbar}M_0c^2t}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \end{aligned} \tag{68}$$

of packets of rapidly varying waves given by the rest energy of the particle, modulated by phase terms depending on the electromagnetic potentials and metric elements, with the group velocity

$$\frac{\mathrm{d}x^j}{\mathrm{d}t} = -\frac{e}{M_0}g^{jk}A_k\left(x^\mu\right). \tag{69}$$

We consider a Fabry-Perot cavity (**Figure 3**).

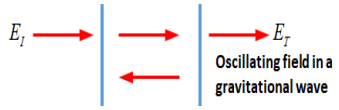

Figure 3: Fabry-Perot cavity in a gravitational wave.

in a gravitational wave,

$$g^{\alpha\beta}g_{\mu\nu,\alpha\beta} = 0. \tag{70}$$

For an incident electromagnetic field

$$E_I = \mathcal{E}_Ie^{\mathrm{i}\left(k_jx-\omega t\right)} \tag{71}$$

of photons with the momentum $\hbar k$ and energy $\hbar\omega$, in resonance with the cavity and with a system of active atoms in this cavity,

$$\begin{aligned} k_j &= -\frac{1}{\hbar}P_j = -\frac{1}{\hbar}\left[-M_0cg_{ij}\dot{x}^i+eA_j\left(x^\mu\right)\right] \simeq \frac{1}{\hbar}g_{ij}M_0c\dot{x}^i \\ \omega &= \frac{e}{\hbar}\left[U\left(x^i\right)+g^{jk}A_k\left(x^\mu\right)\frac{P_j-eA_j\left(x^\mu\right)}{M_0}\right] \simeq \frac{e}{\hbar}U\left(x^i\right), \end{aligned} \tag{72}$$

we obtain an oscillation of the wave vector k_j, proportional to the metric tensor g_{ij} as a solution of the gravitational wave equation (70),

$$g^{\alpha\beta}k_{j,\alpha\beta} = 0, \tag{73}$$

as the atomic resonance frequency and the active electron velocity remain practically unchanged. In this way, by the oscillation of the cavity resonance condition corresponding to the wave vector k_j, an amplitude oscillation of the transmitted wave E_T is obtained.

Quantum Particle in a Central Gravitational Field

The dynamics of a quantum particle in the proper time,

$$\mathrm{d}s^2 \simeq c^2\mathrm{d}t^2 >> \dot{x}^{j^2}, \tag{74}$$

represented by the wave functions (23) and (24) with the Lagrangian (27),

$$\begin{aligned} \psi\left(x^i,t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi\left(P_j,t\right)e^{\frac{i}{\hbar}\left[P_jx^j+M_0c^2\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j}t\right]}\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P_3 \\ \varphi\left(P_j,t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi\left(x^i,t\right)e^{-\frac{i}{\hbar}\left[P_jx^j+M_0c^2\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j}t\right]}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \end{aligned} \tag{75}$$

and the momentum (28),

$$P_{\ j} = -M_0cg_{ij}\dot{x}^i, \tag{76}$$

in a gravitational field is described by the geodesic equation.

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2}=-\Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}. \tag{77}$$

From this equation, we obtain the space acceleration

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}=-c^2\Gamma^i_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta, \tag{78}$$

which with the Christoffel symbol as a function of the metric tensor elements takes the form

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}=-\frac{1}{2}c^2g^{i\lambda}\left(g_{\lambda\alpha,\beta}+g_{\lambda\beta,\alpha}-g_{\alpha\beta,\lambda}\right)\dot{x}^\alpha\dot{x}^\beta. \tag{79}$$

With condition (74) of the proper time, this equation takes a simpler form

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}=-\frac{1}{2}c^2g^{i\lambda}\left(g_{\lambda0,0}+g_{\lambda0,0}-g_{00,\lambda}\right), \tag{80}$$

which for a diagonal metric tensor, is

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}=\frac{1}{2}c^2g^{ii}g_{00,i}. \tag{81}$$

With the Schwarzschild solution (66) for the metric tensor in a central gravitational field, these accelerations take the form of a gradient of the of the gravitational potential,

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}=-c^2g^{ii}\left(\frac{m}{r}\right)_{,i}. \tag{82}$$

We obtain a radial acceleration depending on the mass M_G generating the gravitational field and the gravitation constant G, and null angular accelerations,

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}=-\left(1-\frac{2m}{r}\right)\frac{mc^2}{r^2}=-\left(1-\frac{2GM_G}{c^2r}\right)G\frac{M_G}{r^2} \tag{83}$$
$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=0,\qquad \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}=0.$$

For a radial motion,

$$\frac{\mathrm{d}x^2}{\mathrm{d}s}=\frac{\mathrm{d}x^3}{\mathrm{d}s}=0, \tag{84}$$

from (77) we obtain the geodesic equations

$$\begin{aligned}
&\frac{\mathrm{d}^2x^0}{\mathrm{d}s^2}=-\Gamma^0_{01}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}-\Gamma^0_{10}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s}=-2g^{00}\Gamma_{001}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}\\
&\quad=-g^{00}\left(g_{00,1}+g_{01,0}-g_{01,0}\right)\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}=-g^{00}g_{00,1}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}=-g^{00}\frac{\mathrm{d}g_{00}}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s}\\
&\frac{\mathrm{d}^2x^1}{\mathrm{d}s^2}=-\Gamma^1_{01}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}-\Gamma^1_{10}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s}=-2g^{11}\Gamma_{101}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}\\
&\quad=-g^{11}\left(g_{10,1}+g_{11,0}-g_{10,1}\right)\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}=-g^{11}g_{11,0}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}=-g^{11}\frac{\mathrm{d}g_{11}}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}.
\end{aligned} \tag{85}$$

With the relations

$$g^{00}=\frac{1}{g_{00}},\qquad g^{11}=\frac{1}{g_{11}}, \tag{86}$$

equations (85) take an integrable form

$$\begin{aligned}
&g_{00}\frac{\mathrm{d}^2x^0}{\mathrm{d}s^2}+\frac{\mathrm{d}g_{00}}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s}=0\\
&g_{11}\frac{\mathrm{d}^2x^1}{\mathrm{d}s^2}+\frac{\mathrm{d}g_{11}}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}=0.
\end{aligned} \tag{87}$$

By integrating these equations with the integration constants k_0 and k_1, we obtain

$$\begin{aligned}
&g_{00}v^0=k_0\\
&g_{11}v^1=k_1.
\end{aligned} \tag{88}$$

From with the Schwarzschild solution (66), we obtain the relation

$$g_{00}g_{11}=-1, \tag{89}$$

and, with the first equation (88), the time velocity as a function of the integration constant k_0,

$$v^0=\frac{k_0}{1-\frac{2m}{r}}. \tag{90}$$

At the same time, the fundamental equation (12) takes the form

$$g_{00}v^{0^2}+g_{11}v^{1^2}=1, \tag{91}$$

which, with the first equation (88) and the expressions (89) and (90), leads to the coordinate velocity as a function of the same integration constant k_0,

$$v^1=\left(k_0{}^2-g_{00}\right)^{1/2}=\left(k_0{}^2+\frac{2m}{r}-1\right)^{1/2}. \tag{92}$$

With (90) and (92), we obtain the radial velocity of the particle

$$\frac{\mathrm{d}r}{\mathrm{d}t}=c\frac{\mathrm{d}x^1}{\mathrm{d}x^0}=c\frac{v^1}{v^0}=-\frac{c}{k_0}\left(k_0{}^2+\frac{2m}{r}-1\right)^{1/2}\left(1-\frac{2m}{r}\right). \tag{93}$$

We notice that both expression (83) and (93), of the acceleration and velocity in a central gravitational field, have a singularity for a coordinate r equal to the value

$$r_0=2m, \tag{94}$$

of course, if the mass M_G is sufficiently large to concentrate this mass in a body with a radius

$$r_G<r_0. \tag{95}$$

From the time-space interval with the Schwarzschild metric elements in a gravitational field,

$$ds^2=\left(1-\frac{r_0}{r}\right)c^2dt^2-\left(1-\frac{r_0}{r}\right)^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2, \quad (96)$$

We notice that the frequency of an electromagnetic radiation emitted in a central gravitational field at the coordinate r, as the inverse of the period Δt,

$$\frac{1}{\Delta t}=\sqrt{1-\frac{r_0}{r}}\frac{c}{\Delta s}, \quad (97)$$

becomes null for $r=r_0$ – black hole. From (83) and (93), we notice that a differential element of matter is attracted to the boundary of a black hole, from the outside and also from the inside, without being allowed to pass this barrier, where its acceleration and velocity are null. However, this behavior is not a realistic one, since any matter element is a part of the surrounding matter of the other elements of the quantum particle and, more than that, of the other particle perturbing the gravitational field which is no more constant. The black hole boundary is swept out by gravitational fluctuations, favoring absorption and evaporation processes through the black hole boundary.

Gravitational Dynamics and Spin

We consider a quantum particle described the wave functions

$$\psi\left(x^i,t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi\left(P_j,t\right)e^{\frac{i}{\hbar}\left[P_jx^j-L\left(x^\alpha,\dot{x}^\alpha\right)t\right]}dP_1dP_2dP_3 \quad (98)$$

$$\varphi\left(P_j,t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi\left(x^i,t\right)e^{-\frac{i}{\hbar}\left[P_jx^j-L\left(x^\alpha,\dot{x}^\alpha\right)t\right]}dx^1dx^2dx^3,$$

with the Lagrangian

$$L\left(x^\alpha,\dot{x}^\alpha\right)=-M_0c^2\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}. \quad (99)$$

and the spin (28) in the proper time,

$$P_j=\frac{\partial L}{c\partial\dot{x}^j}=-M_0cg_{ij}\dot{x}^i. \quad (100)$$

From the group velocity in the momentum space with the fundamental equation (12),

$$\frac{d}{dt}P_j=c\dot{P}_j=-M_0c^2g_{ij,k}\dot{x}^k\dot{x}^i-M_0c^2g_{ij}\ddot{x}^i=\frac{\partial L}{\partial x^j} \quad (101)$$

$$=-M_0c^2\frac{\dot{x}^\alpha\dot{x}^\beta g_{\alpha\beta,j}}{2\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}=-\frac{1}{2}M_0c^2g_{\alpha\beta,j}\dot{x}^\alpha\dot{x}^\beta,$$

which in the proper time system of a sufficiently confined distribution of matter, to define a particle, $\dot{x}^i<<\dot{x}^0\simeq 1$, is

$$\ddot{x}^k=\frac{1}{2}g^{kj}g_{00,j}, \quad (102)$$

As a generalization of the geodesic equation (81) for a diagonal metric tensor.

In a gravitational wave,

$$g^{mn}g_{rs,mn}=0, \quad (103)$$

for the metric tensor we can consider a first order solution,

$$g_{\rho\sigma}=u_{\rho\sigma}l_\mu x^\mu, \quad (104)$$

as a function of the polarization vector l_μ satisfying the normalization condition

$$l^\nu l_\nu=0, \quad (105)$$

and the amplitude tensor proportional to the total amplitude u,

$$u_{\rho\sigma}=ug_{\rho\sigma},u^{\rho\sigma}=ug^{\rho\sigma}. \quad (106)$$

The elements of the amplitude tensor are functions of the total amplitude,

$$g_{\mu\rho}u^{\rho\sigma}=u_\mu^\sigma=ug_{\mu\rho}g^{\rho\sigma}=ug_\mu^\sigma, \quad (107)$$

which is

$$u_\mu^\sigma=u\delta_\mu^\sigma. \quad (108)$$

By the index contraction, we obtain

$$u_\mu^\mu=4u, \quad (109)$$

as from the scalar product of equation (108) with the polarization vector we obtain the system of equations

$$l_\sigma u_\mu^\sigma=l_\mu u. \quad (110)$$

With the solution (104) of the gravitational wave (103), the dynamic equation (102)

$$\ddot{x}^k=\frac{1}{2}g^{kj}u_{00}l_j=\frac{1}{2}u_{00}l^k. \quad (111)$$

For a weak gravitational field,

$$g_{00}=1,\ g_{11}=-1,\ g_{22}=-1,\ g_{33}=-1 \quad (112)$$
$$g^{00}=1,\ g^{11}=-1,\ g^{22}=-1,\ g^{33}=-1,$$

from (106) we obtain

$$u_{00}=ug_{00}=u, \quad (113)$$

as the dynamic equation (111) becomes

$$\ddot{x}^k=\frac{1}{2}ul^k, \quad (114)$$

which means an acceleration in the direction x^k, under the action of a gravitational wave with the amplitude u and the polarization l^k. For simplicity, we consider a gravitational wave with the polarization vector,

$$l_0=1,\quad l_1=l_2=0,\quad l_3=-1 \qquad (115)$$
$$l^0=1,\quad l^1=l^2=0,\quad l^3=1,$$

which satisfies the normalization condition (105). With this polarization vector, equations (110) take the form

$$\begin{aligned} u_0^0-u_0^3&=g^{00}u_{00}-g^{33}u_{30}=u_{00}+u_{30}=u\\ u_1^0-u_1^3&=g^{00}u_{01}-g^{33}u_{31}=u_{01}+u_{31}=0\\ u_2^0-u_2^3&=g^{00}u_{02}-g^{33}u_{32}=u_{02}+u_{32}=0\\ u_3^0-u_3^3&=g^{00}u_{03}-g^{33}u_{33}=u_{03}+u_{33}=-u. \end{aligned} \qquad (116)$$

With the metric elements (112) and the expression (109), from the first and the last equations we obtain

$$2u_{03}=-\left(u_{00}+u_{33}\right) \qquad (117)$$

and

$$\underline{u_{00}-u_{33}}=2u=\frac{1}{2}u_\mu^\mu=\frac{1}{2}\left(g^{00}u_{00}+g^{11}u_{11}+g^{22}u_{22}+g^{33}u_{33}\right)=\frac{1}{2}\left(\underline{u_{00}}-u_{11}-u_{22}-\underline{u_{33}}\right), \qquad (118)$$

which is

$$u_{11}+u_{22}=-\left(u_{00}-u_{33}\right)=-2u. \qquad (119)$$

With the second and the third equations (116), and the transformation expression is

$$u^{\alpha\beta}=g^{\alpha\mu}g^{\beta\nu}u_{\mu\nu}=g^{\alpha\alpha}g^{\beta\beta}u_{\alpha\beta}, \qquad (120)$$

we calculate the scalar

$$\begin{aligned} I_u&=u_{\alpha\beta}u^{\alpha\beta}-2u^2\\ &=u_{00}{}^2+u_{11}{}^2+u_{22}{}^2+u_{33}{}^2-2\underline{u_{01}}{}^2-2\underline{\underline{u_{02}}}{}^2-\underline{\underline{\underline{2u_{03}}}}{}^2+2u_{12}{}^2+2\underline{u_{13}}{}^2+2\underline{\underline{u_{23}}}{}^2-\frac{1}{2}\left(u_{00}-u_{33}\right)^2\\ &=u_{11}{}^2+u_{22}{}^2+2u_{12}{}^2+\underbrace{\underline{\underline{u_{00}}}{}^2+\underline{\underline{u_{33}}}{}^2-\frac{1}{2}\left(u_{00}{}^2+u_{33}{}^2\right)}-u_{00}u_{33}-\underbrace{\frac{1}{2}\left(u_{00}-u_{33}\right)^2}\\ &=u_{11}{}^2+u_{22}{}^2+2u_{12}{}^2=\frac{1}{2}\left[\underline{\left(u_{11}-u_{22}\right)^2}+\underbrace{\left(u_{11}+u_{22}\right)^2}_{\underline{\underline{4u^2}}}\right]+\underline{2u_{12}}{}^2=I_0+2u^2, \end{aligned} \qquad (121)$$

with the amplitude term $2u^2$ as an intensity of the gravitational wave, and a term

$$I_0=\frac{1}{2}\left(u_{11}-u_{22}\right)^2+2u_{12}{}^2, \qquad (122)$$

as an invariant for the rotations of the amplitude tensor $u_{\mu\nu}$ in the plane of the coordinates x^1,x^2. For a rotation with $-\frac{\pi}{2}$ of a vector $A=\left(A^1,A^2\right)$ in this plane,

$$\begin{aligned} RA^1&=A^2\\ RA^2&=-A^1, \end{aligned} \qquad (123)$$

we obtain the equation

$$R^2A^1=-A^1, \qquad (124)$$

with the eigenvalue

$$R=\pm\mathrm{i}. \qquad (125)$$

For a second order tensor, these rotation transformations take the form

$$\begin{aligned} Ru_{11}&=u_{12}=u_{21}=\frac{1}{2}\left(u_{12}+u_{21}\right)\\ Ru_{22}&=-u_{21}=-u_{12}=-\frac{1}{2}\left(u_{12}+u_{21}\right)\\ Ru_{21}&=Ru_{12}=\frac{1}{2}\left(u_{22}-u_{11}\right). \end{aligned} \qquad (126)$$

From the first two equations, we obtain

$$\begin{aligned} R\left(u_{11}+u_{22}\right)&=0\\ R\left(u_{11}-u_{22}\right)&=2u_{12}. \end{aligned} \qquad (127)$$

From the second equation with the third equation (126), we obtain the equation

$$R^2\left(u_{11}-u_{22}\right)=2Ru_{12}=u_{22}-u_{11} \qquad (128)$$

with the same eigenvalue (125), those transformations, by a rotation R of the first term of the invariant (122), we obtain the second term,

$$\frac{1}{2}\left[R\left(u_{11}-u_{22}\right)\right]^2=\frac{1}{2}\left(2u_{12}\right)^2=2u_{12}{}^2, \qquad (129)$$

as by a rotation R of the second term, we obtain the first term,

$$2\left(Ru_{12}\right)^2 = 2\left[\frac{1}{2}\left(u_{22}-u_{11}\right)\right]^2 = \frac{1}{2}\left(u_{11}-u_{22}\right)^2, \tag{130}$$

which means that by a rotation with an angle $-\frac{\pi}{2}$, $R = R^{-\pi/2}$ the terms of the invariant (122) transform one another. A rotation operator of a vector $A^i\left(\vec{r}\right)$ with an angle $\delta\vec{\alpha}$,

$$\begin{aligned} R^{\delta\vec{\alpha}}A^i\left(\vec{r}\right) &= A^i\left(\vec{r}+\delta\vec{\alpha}\times\vec{r}\right) = A^i\left(\vec{r}\right)+\delta\vec{\alpha}\times\vec{r}\cdot\frac{\partial}{\partial\vec{r}}A^i\left(\vec{r}\right) \\ &= A^i\left(\vec{r}\right)+\delta\vec{\alpha}\cdot\vec{r}\times\frac{\partial}{\partial\vec{r}}A^i\left(\vec{r}\right) = A^i\left(\vec{r}\right)+\mathrm{i}\delta\vec{\alpha}\times\vec{S}A^i\left(\vec{r}\right) \\ &= e^{\mathrm{i}\vec{S}\delta\vec{\alpha}}A^i\left(\vec{r}\right), \end{aligned} \tag{131}$$

as a function of the angular momentum operator

$$\vec{S} = -\mathrm{i}\vec{r}\times\frac{\partial}{\partial\vec{r}}, \tag{132}$$

is of the form

$$R^{\delta\vec{\alpha}} = e^{\mathrm{i}\vec{S}\delta\vec{\alpha}}. \tag{133}$$

We notice that the rotation with an angle π of a vector is equivalent with an inversion,

$$R^{\pi}A^i\left(\vec{r}\right) = e^{\mathrm{i}S\pi}A^i\left(\vec{r}\right) = -A^i\left(\vec{r}\right), \tag{134}$$

which means that the proper angular momentum of a vector, we call spin, is $S=1$. From the invariance of the scalar product of the amplitude tensor u_{ij} of a gravitational wave with two vectors A^i and B^j with the spin 1,

$$\begin{aligned} \text{Scalar} = u_{ij}\left(x,y\right)A^i\left(x,y\right)B^j\left(x,y\right) &= \left[R^{\delta\vec{\alpha}}u_{ij}\left(x,y\right)\right]\left[R^{\delta\vec{\alpha}}A^i\left(x,y\right)\right]\left[R^{\delta\vec{\alpha}}B^j\left(x,y\right)\right] \\ &= \left[e^{\mathrm{i}\vec{S}_1\delta\vec{\alpha}}u_{ij}\left(x,y\right)\right]\left[e^{\mathrm{i}\vec{S}\delta\vec{\alpha}}A^i\left(x,y\right)\right]\left[e^{\mathrm{i}\vec{S}\delta\vec{\alpha}}B^j\left(x,y\right)\right] \\ &= \left[e^{\mathrm{i}\vec{S}_1\delta\vec{\alpha}}u_{ij}\left(x,y\right)\right]\left[e^{\mathrm{i}\delta\vec{\alpha}}A^i\left(x,y\right)\right]\left[e^{\mathrm{i}\delta\vec{\alpha}}B^j\left(x,y\right)\right], \end{aligned} \tag{135}$$

we obtain the rotation operator of a tensor,

$$R^{\delta\vec{\alpha}}u_{ij}\left(x,y\right) = e^{\mathrm{i}\vec{S}_1\delta\vec{\alpha}}u_{ij}\left(x,y\right) = e^{-2\mathrm{i}\vec{S}_1\delta\vec{\alpha}}u_{ij}\left(x,y\right) = e^{-2\mathrm{i}\delta\vec{\alpha}}u_{ij}\left(x,y\right), \tag{136}$$

which means a graviton spin $S_t = 2$. This spin describes the rotation of the intrinsic spatial components of a quantum particle described by the space coordinates x^i and the invariant (12), and the dynamic equation (102), in a gravitational field described by the metric tensor g_{00} as a potential. As we have shown by the expressions (129-130), for a rotation of the two terms of the invariant of the spatial components change their values by a rotation with $\frac{\pi}{2}$, which means that they take the same value by a complete rotation with 2π - spin 2.

For a rotation with a differential angle $\vec{\alpha}$ of the extrinsic component of a quantum particle with a distribution function $\psi\left(\vec{r},t\right)$,

$$\begin{aligned} R_{\delta\vec{\alpha}}\psi\left(\vec{r}\right) &= \psi\left(\vec{r}+\delta\vec{\alpha}\times\vec{r}\right) = \psi\left(\vec{r}\right)+\delta\vec{\alpha}\times\vec{r}\frac{\partial}{\partial\vec{r}}\psi\left(\vec{r}\right) \\ &= \psi\left(\vec{r}\right)+\delta\vec{\alpha}\cdot\vec{r}\times\underset{\mathrm{i}\vec{S}}{\frac{\partial}{\partial\vec{r}}}\psi\left(\vec{r}\right), \end{aligned} \tag{137}$$

we obtain the rotation operator as a function of the proper rotation angular momentum we call particle spin,

$$R_{\vec{\alpha}} = e^{\mathrm{i}\vec{S}\vec{\alpha}}. \tag{138}$$

For a two-particle state,

$$\psi_{i_1i_2}\left(\vec{r}_1,\vec{r}_2\right) = \left\langle\vec{r}_1,\vec{r}_2\middle|i_1,i_2\right\rangle, \tag{139}$$

by applying two times the inversion operator,

$$\left\langle\vec{r}_1,\vec{r}_2\middle|i_1,i_2\right\rangle = I\left\langle\vec{r}_2,\vec{r}_1\middle|i_1,i_2\right\rangle = I^2\left\langle\vec{r}_1,\vec{r}_2\middle|i_1,i_2\right\rangle, \tag{140}$$

we obtain $I^2=1$ which correspond to two types of particles: Fermions with $I=-1$ and Bosons with $I=1$. Taking into account that an inversion is equivalent with a double rotation of two differential elements of matter, with the two rotation operators

$$R_{\pi}^{(1)} = R_{\pi}^{(2)} = e^{\mathrm{i}\pi S}, \tag{141}$$

for Fermions we obtain:

$$I = R_{\pi}^{(1)}R_{\pi}^{(2)} = e^{2\mathrm{i}\pi S} = -1, \tag{142}$$

which means a half-integer spin, $S = \frac{1}{2},\frac{3}{2},\ldots$ as for Bosons

$$I = R_{\pi}^{(1)}R_{\pi}^{(2)} = e^{2\mathrm{i}\pi S} = 1, \tag{143}$$

which means an integer spin $S = 1,2,\ldots$.

For the gravitational wave equation (103), we can consider also a second order solution,

$$g_{\rho\sigma} = u_{\rho\sigma}l_{\mu\nu}x^{\mu}x^{\nu}, \tag{144}$$

with an amplitude tensor $u_{\rho\sigma}$, and a polarization tensor $l_{\mu\nu}$, which according to (103), satisfies the normalization equation

$$l_{\nu}^{\nu} = 0. \tag{145}$$

From (102), we obtain the dynamic equation

$$\ddot{x}^k = \frac{1}{2}g^{kj}g_{00,j} = \frac{1}{2}g^{kj}u_{00}l_{\mu j}x^{\nu} = \frac{1}{2}u_{00}l_{\mu}^{k}x^{\mu}. \tag{146}$$

For a gravitational field oscillating in the direction x^1,

$$l_0^0 = 1, \quad l_1^1 = -1, \quad l_2^2 = l_3^3 = 0, \qquad l_0^0 + l_1^1 + l_2^2 + l_3^3 = 0, \tag{147}$$

this dynamic equation takes the form of a harmonic oscillator with the coordinate x^1, which, with (113) is

$$\ddot{x}^1 = -\frac{1}{2}ux^1. \tag{148}$$

Graviton Hamiltonian Tensor

The gravitational field is an effect of the curvature of the time-space hypersurface $\{x^\nu\}$ in the total space $\{y^n\}$, according to the expressions (2)-(3). We define the graviton action as an integral of the total curvature R with the scalar density $\sqrt{\ } = \sqrt{-g}$, on its time-space volume,

$$I_g = \int R\sqrt{\ }\,\mathrm{d}^4x. \tag{149}$$

For the total curvature R as a function of the Ricci tensor (A.5), we consider the expression

$$R = g^{\mu\nu}R_{\mu\nu} = R^* - L, \tag{150}$$

with the terms

$$\begin{aligned} R^* &= g^{\mu\nu}\left(\Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha}\right) \\ L &= g^{\mu\nu}\left(\Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu}\right). \end{aligned} \tag{151}$$

With these terms, the graviton action (149) takes the form

$$I_g = \int\left(R^* - L\right)\sqrt{\ }\,\mathrm{d}^4x, \tag{152}$$

where the first term can be integrated by parts,

$$R^*\sqrt{\ } = \left(g^{\mu\nu}\Gamma^\alpha_{\mu\alpha}\sqrt{\ }\right)_{,\nu} - \left(g^{\mu\nu}\Gamma^\alpha_{\mu\nu}\sqrt{\ }\right)_{,\alpha} - \left(g^{\mu\nu}\sqrt{\ }\right)_{,\nu}\Gamma^\alpha_{\mu\alpha} + \left(g^{\mu\nu}\sqrt{\ }\right)_{,\alpha}\Gamma^\alpha_{\mu\nu}. \tag{153}$$

Since, by the action definition, the total derivatives have no contribution in the integral (152) with fixed integration limits, we take

$$R^*\sqrt{\ } = \left(g^{\mu\nu}\sqrt{\ }\right)_{,\alpha}\Gamma^\alpha_{\mu\nu} - \left(g^{\mu\nu}\sqrt{\ }\right)_{,\nu}\Gamma^\alpha_{\mu\alpha}. \tag{154}$$

With the formulas

$$\begin{aligned} g^{\mu\nu}{}_{,\alpha} &= -g^{\mu\rho}g^{\nu\sigma}g_{\rho\sigma,\alpha} \\ g_{\rho\sigma,\alpha} &= \Gamma_{\rho\sigma\alpha} + \Gamma_{\sigma\rho\alpha} \\ \sqrt{\ }_{,\alpha} &= \Gamma^\sigma_{\alpha\sigma}\sqrt{\ }, \end{aligned} \tag{155}$$

for the two terms of (154), we obtain the expressions

$$\begin{aligned} \left(g^{\mu\nu}\sqrt{\ }\right)_{,\alpha} &= g^{\mu\nu}{}_{,\alpha}\sqrt{\ } + g^{\mu\nu}\sqrt{\ }_{,\alpha} \\ &= -g^{\mu\rho}g^{\nu\sigma}\left(\Gamma_{\rho\sigma\alpha} + \Gamma_{\sigma\rho\alpha}\right)\sqrt{\ } + g^{\mu\nu}\Gamma^\sigma_{\alpha\sigma}\sqrt{\ } \\ &= \left(-g^{\nu\sigma}\Gamma^\mu_{\sigma\alpha} - g^{\mu\rho}\Gamma^\nu_{\rho\alpha} + g^{\mu\nu}\Gamma^\sigma_{\alpha\sigma}\right)\sqrt{\ }, \\ \left(g^{\mu\nu}\sqrt{\ }\right)_{,\nu} &= \left(-g^{\nu\sigma}\Gamma^\mu_{\sigma\nu} - \underline{g^{\mu\rho}\Gamma^\nu_{\rho\nu}} + \underline{g^{\mu\nu}\Gamma^\sigma_{\nu\sigma}}\right)\sqrt{\ } \\ &= -g^{\nu\sigma}\Gamma^\mu_{\sigma\nu}\sqrt{\ }. \end{aligned} \tag{156}$$

With these expressions, from (154) with the second expression (151) and the symmetry relations of the Christoffel symbols, we obtain

$$\begin{aligned} R^*\sqrt{\ } &= \left(\underline{-g^{\nu\sigma}\Gamma^\mu_{\sigma\alpha} - g^{\mu\rho}\Gamma^\nu_{\rho\alpha}} + g^{\mu\nu}\Gamma^\sigma_{\alpha\sigma}\right)\underline{\Gamma^\alpha_{\mu\nu}}\sqrt{\ } + g^{\nu\sigma}\Gamma^\mu_{\sigma\nu}\,\Gamma^\alpha_{\mu\alpha}\sqrt{\ } \\ &= \left(-2g^{\nu\sigma}\Gamma^\mu_{\sigma\alpha} + \underline{g^{\mu\nu}\Gamma^\sigma_{\alpha\sigma}}\right)\Gamma^\alpha_{\mu\nu}\sqrt{\ } + \underline{g^{\nu\sigma}\Gamma^\mu_{\sigma\nu}\,\Gamma^\alpha_{\mu\alpha}\sqrt{\ }} \\ &= \left(2g^{\mu\nu}\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\alpha\sigma} - 2g^{\nu\sigma}\Gamma^\mu_{\sigma\alpha}\Gamma^\alpha_{\mu\nu}\right)\sqrt{\ } \\ &= \left(2g^{\mu\nu}\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\alpha\sigma} - 2g^{\nu\mu}\Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\sigma\nu}\right)\sqrt{\ } \\ &= 2L\sqrt{\ }. \end{aligned} \tag{157}$$

In this way, for the graviton action (149),

$$\begin{aligned} I_g &= \int R\sqrt{\ }\,\mathrm{d}^4x = \int\left(R^* - L\right)\sqrt{\ }\,\mathrm{d}^4x = \int L\sqrt{\ }\,\mathrm{d}^4x \\ &= \int \mathcal{L}\mathrm{d}^4x = \int \mathrm{d}x^0\int \mathcal{L}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \end{aligned} \tag{158}$$

we obtain the Lagrangian density

$$\begin{aligned} \mathcal{L} = L\sqrt{\ } &= \left(\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}\right)g^{\mu\nu}\sqrt{\ } \\ &= g^{\alpha\rho}g^{\beta\sigma}\left(\Gamma_{\rho\mu\nu}\Gamma_{\sigma\alpha\beta} - \Gamma_{\rho\mu\beta}\Gamma_{\sigma\alpha\nu}\right)g^{\mu\nu}\sqrt{\ }. \end{aligned} \tag{159}$$

Since the second kind Christoffel symbols are explicit expressions of the metric elements $g^{\mu\nu}$, and their derivative $g_{\mu\nu,\sigma}$, we consider these elements as gravitation coordinates and gravitation velocities in the graviton action (158). The variation of this action takes the form

$$\begin{aligned} \delta I_g &= \int \delta\mathcal{L}\mathrm{d}^4x = \int\left(\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta} + \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\nu}}\delta g_{\alpha\beta,\nu}\right)\mathrm{d}^4x \\ &= \int\left[\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}} - \left(\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\nu}}\right)_{,\nu}\right]\delta g_{\alpha\beta}\mathrm{d}^4x = 0, \end{aligned} \tag{160}$$

with the graviton Lagrangian density variation

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta} + \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\nu}}\delta g_{\alpha\beta,\nu}, \tag{161}$$

and the Lagrange equations

$$\left(\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\nu}}\right)_{,\nu} = \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}}. \tag{162}$$

With the graviton momentum tensor

$$P^{\alpha\beta\nu} = \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\nu}}, \tag{163}$$

we define the graviton Hamiltonian tensor

$$t^\nu_\mu\sqrt{\ } = P^{\alpha\beta\nu}g_{\alpha\beta,\mu} - g^\nu_\mu\mathcal{L} = \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\nu}}g_{\alpha\beta,\mu} - g^\nu_\mu\mathcal{L}. \tag{164}$$

Since the first Christoffel symbols are functions of the derivatives of the metric tensor elements, from the graviton Lagrangian density (159) we obtain

$$\frac{\partial \mathcal{L}}{\partial\left(g^{\mu\nu}\sqrt{-}\right)}=\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}-\Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}, \tag{165}$$

as, with (B.7)

$$\frac{\partial \mathcal{L}}{\partial\left(g^{\alpha\beta}\sqrt{-}\right)_{,\nu}}=\Gamma^{\nu}_{\alpha\beta}-g^{\nu}_{\beta}\Gamma^{\sigma}_{\alpha\sigma}. \tag{166}$$

On the other hand, with a change of coordinates

$$Q^{m}{}_{,\mu}=\frac{\partial Q^{m}}{\partial q^{n}}q^{n}{}_{,\mu} \tag{167}$$

the first term of the graviton Hamiltonian (164) is not modified,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial q^{n}{}_{,\nu}}q^{n}{}_{,\mu}&=\frac{\partial \mathcal{L}}{\partial Q^{m}{}_{,\sigma}}\frac{\partial Q^{m}{}_{,\sigma}}{\partial q^{n}{}_{,\nu}}q^{n}{}_{,\mu}=\frac{\partial \mathcal{L}}{\partial Q^{m}{}_{,\sigma}}\frac{\partial\frac{\partial Q^{m}}{\partial q^{n}}q^{n}{}_{,\sigma}}{\partial q^{n}{}_{,\nu}}q^{n}{}_{,\mu}=\frac{\partial \mathcal{L}}{\partial Q^{m}{}_{,\sigma}}\frac{\partial Q^{m}}{\partial q^{n}}\frac{\partial q^{n}{}_{,\sigma}}{\partial q^{n}{}_{,\nu}}q^{n}{}_{,\mu}\\&=\frac{\partial \mathcal{L}}{\partial Q^{m}{}_{,\sigma}}\frac{\partial Q^{m}}{\partial q^{n}}g^{\nu}_{\sigma}q^{n}{}_{,\mu}=\frac{\partial \mathcal{L}}{\partial Q^{m}{}_{,\nu}}\frac{\partial Q^{m}}{\partial q^{n}}q^{n}{}_{,\mu}=\frac{\partial \mathcal{L}}{\partial Q^{m}{}_{,\nu}}Q^{m}{}_{,\mu}.\end{aligned} \tag{168}$$

which means that the graviton Hamiltonian (164) is of the form.

$$t^{\nu}_{\mu}\sqrt{-}=\left(\Gamma^{\nu}_{\alpha\beta}-g^{\nu}_{\beta}\Gamma^{\sigma}_{\alpha\sigma}\right)\left(g^{\alpha\beta}\sqrt{-}\right)_{,\mu}-g^{\nu}_{\mu}\mathcal{L}. \tag{169}$$

This tensor takes a simpler explicit form for the first-order metric tensor (104) in a gravitational wave. In this case,

$$\underline{g^{\alpha\beta}{}_{,\mu}}=-g^{\alpha\rho}g^{\beta\sigma}\underline{g_{\rho\sigma,\mu}}=-g^{\alpha\rho}g^{\beta\sigma}u_{\rho\sigma}l_{\mu}=-u^{\alpha\beta}l_{\mu}, \tag{170}$$

as, with (107), by an index contraction,

$$\underline{\underline{\sqrt{-}_{,\mu}}}=\frac{1}{2\sqrt{-}}\left(-g\right)_{,\mu}=-\frac{1}{2\sqrt{-}}g\underline{g^{\alpha\beta}}\underline{g_{\alpha\beta,\mu}}=\frac{1}{2}\sqrt{-}g^{\alpha\beta}\underline{g_{\alpha\beta,\mu}}=\frac{1}{2}\sqrt{-}g^{\alpha\beta}\left(u_{\alpha\beta}l_{\mu}\right)=2\sqrt{-}ul_{\mu}. \tag{171}$$

We obtain

$$\begin{aligned}\left(g^{\alpha\beta}\sqrt{-}\right)_{,\mu}&=\underline{g^{\alpha\beta}{}_{,\mu}}\sqrt{-}+g^{\alpha\beta}\underline{\underline{\sqrt{-}_{,\mu}}}=-u^{\alpha\beta}l_{\mu}\sqrt{-}+g^{\alpha\beta}2\sqrt{-}ul_{\mu}\\&=-\left(u^{\alpha\beta}-2g^{\alpha\beta}u\right)\sqrt{-}l_{\mu}=ug^{\alpha\beta}\sqrt{-}l_{\mu}.\end{aligned} \tag{172}$$

At the same time, the first Christoffel symbol of the graviton Hamiltonian tensor (169),

$$\Gamma^{\nu}_{\alpha\beta}=g^{\nu\lambda}\Gamma_{\lambda\alpha\beta}, \tag{173}$$

with

$$\Gamma_{\lambda\alpha\beta}=\frac{1}{2}\left(g_{\lambda\alpha,\beta}+g_{\lambda\beta,\alpha}-g_{\alpha\beta,\lambda}\right)=\frac{1}{2}\left(u_{\lambda\alpha}l_{\beta}+u_{\lambda\beta}l_{\alpha}-u_{\alpha\beta}l_{\lambda}\right), \tag{174}$$

is

$$\begin{aligned}\Gamma^{\nu}_{\alpha\beta}&=g^{\nu\lambda}\frac{1}{2}\left(u_{\lambda\alpha}l_{\beta}+u_{\lambda\beta}l_{\alpha}-u_{\alpha\beta}l_{\lambda}\right)=\frac{1}{2}\left(u^{\nu}_{\alpha}l_{\beta}+u^{\nu}_{\beta}l_{\alpha}-u_{\alpha\beta}l^{\nu}\right)\\&=\frac{1}{2}\left(g^{\nu}_{\alpha}l_{\beta}+g^{\nu}_{\beta}l_{\alpha}-g_{\alpha\beta}l^{\nu}\right)u,\end{aligned} \tag{175}$$

as the second Christoffel symbol of the graviton Hamiltonian (169) is

$$\begin{aligned}\Gamma^{\sigma}_{\alpha\sigma}&=g^{\sigma\lambda}\Gamma_{\lambda\alpha\sigma}=g^{\sigma\lambda}\frac{1}{2}\left(\underline{g_{\lambda\alpha,\sigma}}+g_{\lambda\sigma,\alpha}-\underline{g_{\alpha\sigma,\lambda}}\right)=\frac{1}{2}g^{\sigma\lambda}g_{\lambda\sigma,\alpha}\\&=\frac{1}{2}g^{-1}g_{,\alpha}=\frac{-g_{,\alpha}}{2\sqrt{-g}}\frac{1}{\sqrt{-g}}=\frac{\sqrt{-}_{,\alpha}}{\sqrt{-}}=\left(\ln\sqrt{-}\right)_{,\alpha}=2ul_{\alpha}.\end{aligned} \tag{176}$$

With (172), (175) and (176), the first term of the graviton Hamiltonian tensor (169) is

$$\begin{aligned}\left(\Gamma^{\nu}_{\alpha\beta}-g^{\nu}_{\beta}\Gamma^{\sigma}_{\alpha\sigma}\right)\left(g^{\alpha\beta}\sqrt{-}\right)_{,\mu}&=\left[\frac{1}{2}\left(g^{\nu}_{\alpha}l_{\beta}+g^{\nu}_{\beta}l_{\alpha}-g_{\alpha\beta}l^{\nu}\right)u-g^{\nu}_{\beta}2ul_{\alpha}\right]ug^{\alpha\beta}\sqrt{-}l_{\mu}\\&=\frac{1}{2}\left(g^{\nu\beta}l_{\beta}+g^{\nu\alpha}l_{\alpha}-4l^{\nu}\right)u^{2}\sqrt{-}l_{\mu}-2g^{\nu\alpha}l_{\alpha}u^{2}\sqrt{-}l_{\mu}\\&=-2u^{2}\sqrt{-}l^{\nu}l_{\mu}-2u^{2}\sqrt{-}l^{\nu}l_{\mu}=-4u^{2}\sqrt{-}l^{\nu}l_{\mu},\end{aligned} \tag{177}$$

as the second term, proportional to the graviton Lagrangian density (159),

$$\mathcal{L}=L\sqrt{-}=\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}-\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu}\right)g^{\mu\nu}\sqrt{-}, \tag{178}$$

according to (C.7), is null

$$\mathcal{L}=-2u^{2}l^{\alpha}l_{\alpha}\sqrt{-}=0. \tag{179}$$

With (177) and (179), the graviton Hamiltonian tensor (169) takes the form

$$t^{\nu}_{\mu}\sqrt{-}=-4u^{2}\sqrt{-}l^{\nu}l_{\mu}, \tag{180}$$

with the null trace,

$$\mathrm{Tr}\left\{t^{\nu}_{\mu}\sqrt{-}\right\}=t^{\nu}_{\nu}\sqrt{-}=0, \tag{181}$$

Which means that, for a first order metric tensor in a gravitational wave, the total curvature energy on the four coordinates is null, which means that, in agreement with (104)-(105), the energy on the time coordinate and the energies on the spatial coordinates cancel one another. From (148)-(150), we notice that a graviton, as the time-space component of a quantum particle, has non-zero stretches on the four time-space coordinates, we call particle uncertainties, as the wave function stretches are called in the conventional quantum mechanics.

Conclusions

We showed that the relativistic dynamics of the matter can be understood only as an evolution of quantum particles: according to the general theory of relativity, any acceleration of a matter differential element, induced by an external (non-gravitational) field is perpendicular to the velocity of this element; this means that any matter distribution can be described by a Fourier series expansion of plane waves perpendicular to the velocity. If the time-dependent phases are considered as functions of coordinates and velocities, with wave vectors as velocity derivatives of the coefficients of these time dependent phases, we obtain the group velocity of the inverse Fourier transform as a Lagrange equation; this means that the time-dependent phase coefficient is a relativistic Lagrangian which, unavoidably, is proportional to a mass M_0 as a characteristic of the matter dynamics; this means that the total mass M as an integral of the matter density described by this Fourier expansion is equal to the mass M_0 as a characteristic of the matter dynamics – the matter quantization rule.

We showed that, in the proper time, the group velocity of the wave-packet describing a quantum particle is equal to the velocity of any

differential element of matter as a component of this particle – the wave-packet of a free quantum particle is not dispersing during its propagation, contrary to a wave-packet obtained as a solution of the conventional Schrödinger equation. We obtained a relativistic invariant of the matter density, as the product of this density with the scalar density as the square root of minus the determinant of the metric tensor.

A relativistic conservation equation is obtained for the matter flow four-vector. For a quantum particle interacting with an electromagnetic field in a system of curvilinear coordinates, a relativistic Hamiltonian depending on the metric elements is obtained. For an electromagnetic wave interacting with a matter system in a gravitational wave, we obtained an oscillation of the electromagnetic wave vector induced by the gravitational wave; we found a description of a gravitational wave as an oscillation of the transmission coefficient of an active Fabry-Perot cavity.

For a quantum particle, with the Schwarzschild solution for a central gravitational field, we obtained the acceleration as the Newton gravitational acceleration with a relativistic factor, and the radial velocity, which, at the boundary of a black hole, become null – no matter element can pass the boundary of a black hole in the approximation of a constant gravitational field.

However, in the realistic cases, absorption and evaporation processes arise at the boundary of a black hole, by two effects: **1)** A quantum effect, as the gravitational field applied to a matter differential element of a quantum particle is perturbed by the other matter elements of this particle, and **2)** A statistical effect, by the gravitational fluctuations induced by the neighboring quantum particles. For a quantum particle in a gravitational wave, in the first order approximation of the metric tensor, we found an acceleration under the action of this wave, as in the second-order approximation, a quantum particle vibrates under the action of this wave.

We found that the space coordinate intervals of a quantum particle rotate with a spin 2, we call graviton spin, as the amplitude elements of the metric tensor define an invariant of this rotation. At the same time, the matter contained in the space volume of a quantum particle rotate with a half-integer spin for Fermions and an integer spin for Bosons – the spin-statistics relation. For the curvature of the time-space intervals occupied by a quantum particle, acting as a gravitational field on this particle, we defined a graviton action. For this action, we found a graviton Lagrangian, depending on the metric elements, as gravitational coordinates, and the coordinate derivatives of these elements, as gravitational velocities.

A Lagrange equation is found for these gravitational coordinates and velocities. As a function of the graviton momenta, as derivatives of the graviton Lagrangian with the gravitational velocities, the gravitational velocities, and the graviton Lagrangian density, we defined a graviton Hamiltonian tensor. This graviton Hamiltonian is explicitly calculated for a gravitational wave in the first-order approximation of the metric tensor. In this case, the graviton energies on the coordinates, as the diagonal elements of the graviton Hamiltonian tensor, cancel one another, as the trace of this tensor is null. We found that, the graviton action, as an integral of the time-space curvature on the time-space coordinates, being a non-zero quantity, a quantum particle has non-zero dimensions on the four time-space coordinates – as in quantum mechanics we call these dimensions uncertainties.

References

1. Pagels HR. The cosmic code-quantum physics as the language of nature (1983) Bantam Books, United States.
2. De Broglie L. Theory of quantification in the new mechanics (1932) United Kingdom.
3. Heisenberg W. The physical principles of the quantum theory (1949) Dover Publications, United states.
4. Dirac PAM. The principles of quantum mechanics (1958) Oxford University Press, United Kingdom.
5. Schiff LI. Quantum mechanics (1955) McGraw-Hill United States.
6. Feynman RP, Leighton RB and Sands M. The feynman lectures on physics (1963) Addison-Wesley Publishing Company, United States.
7. Bojowald M. Once before time. A whole story of the universe (2009) Vintage Publisher, United States.
8. Carroll S. Even physicists don't understand quantum mechanics - worse, they don't want to understand it (2019) New York Times, United States.
9. Popper KR. The logic of scientific discovery (1975) Hutchinson and Co LTD, United Kingdom.
10. Brody T. The Philosophy behind Physics (1993), Heidelberg: Springer-Verlag, United Kingdom.
11. Wilber K. Quantum questions-mystical writings of the world's great physicists (2001) Shambhala publications, United States.
12. Hall MJW and Reginatto M. Ensembles on configuration space-classical, quantum, and beyond fundamental theories of physics (2016) Springer, United Kingdom.
13. Barret R, Delsanto PP and Tartaglia A. Physics: The ultimate adventure (2016) Springer, United Kingdom.
14. Plotinsky A. The principles of quantum theory, from planck's quanta to the higgs boson-the nature of quantum reality and the spirit of copenhagen (2016) Springer, United Kingdom.
15. Rowlands P. The foundations of physical law (2015) World Scientific, Singapore.
16. Baggott J. Origins-The scientific story of creation (2015) Oxford University Press, United Kingdom.
17. Cox B and Forshaw J. The quantum universe (2012) Da Capo Press, United States.
18. Smolin L. The trouble with physics-the rise of string theory, the fall of a science and what comes next (2006) Allen Lane an imprint of Penguin Books, United Kingdom.
19. Penrose R. The Road to reality (2004) Vintage Books, United Kingdom.
20. Stefanescu E. The relativistic dynamics as a quantum effect (2014) J Bas App Res Int 1: 13.
21. Stefanescu E. Open quantum physics and environmental heat conversion into usable energy (2014) Bentham Science Publishers, United States.
22. Stefanescu E. Theory of relativity and quantum mechanics as complementary parts of a unitary theory (2019) Int J Rec Sci Res 10: 33304-33319. http://dx.doi.org/10.24327/ijrsr.2019.1006.3647
23. Dirac PAM. General theory of relativity (1975) John Wiley and Sons, United States.
24. Mentor KA and Ramšak A. Aharonov-Bohm effect (2011), Seminar 4, Department of physics, University of Ljubljana, Slovenia.

Appendix A

We consider the curvature tensor $R^{\alpha}_{\nu\rho\sigma}$ defined by an arbitrary vector A_{μ} in a curved times-space system [23],

$$A_{\mu:\nu:\rho} - A_{\mu:\rho:\nu} = A_{\alpha} R^{\alpha}_{\mu\nu\rho} . \tag{A.1}$$

The two terms of this expression have the explicit forms,

$$\begin{aligned} A_{\mu:\nu:\rho} &= A_{\mu:\nu,\rho} - \Gamma^{\beta}_{\mu\rho} A_{\beta:\nu} - \Gamma^{\beta}_{\nu\rho} A_{\mu:\beta} \\ &= \left(A_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu} A_{\alpha}\right)_{,\rho} - \Gamma^{\beta}_{\mu\rho}\left(A_{\beta,\nu} - \Gamma^{\alpha}_{\beta\nu} A_{\alpha}\right) - \Gamma^{\beta}_{\nu\rho}\left(A_{\mu,\beta} - \Gamma^{\alpha}_{\mu\beta} A_{\alpha}\right) \\ &= A_{\mu,\nu,\rho} - \underline{\Gamma^{\alpha}_{\mu\nu} A_{\alpha,\rho}} - \underline{\underline{\Gamma^{\beta}_{\mu\rho} A_{\beta,\nu}}} - \underline{\underline{\Gamma^{\beta}_{\nu\rho} A_{\mu,\beta}}} - \left(\Gamma^{\alpha}_{\mu\nu,\rho} - \Gamma^{\beta}_{\mu\rho}\Gamma^{\alpha}_{\beta\nu} - \underline{\underline{\Gamma^{\beta}_{\nu\rho}\Gamma^{\alpha}_{\mu\beta}}}\right) A_{\alpha} , \end{aligned} \tag{A.2}$$

and

$$A_{\mu:\rho:\nu} = A_{\mu,\rho,\nu} - \underline{\underline{\Gamma^{\alpha}_{\mu\rho} A_{\alpha,\nu}}} - \underline{\Gamma^{\beta}_{\mu\nu} A_{\beta,\rho}} - \underline{\underline{\Gamma^{\beta}_{\rho\nu} A_{\mu,\beta}}} - \left(\Gamma^{\alpha}_{\mu\rho,\nu} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\rho} - \underline{\underline{\Gamma^{\beta}_{\rho\nu}\Gamma^{\alpha}_{\mu\beta}}}\right) A_{\alpha} . \tag{A.3}$$

With the symmetry relations of the Christoffel symbols, for the curvature tensor we obtain the expression

$$R^{\alpha}_{\mu\nu\rho} = \Gamma^{\alpha}_{\mu\rho,\nu} - \Gamma^{\alpha}_{\mu\nu,\rho} + \Gamma^{\beta}_{\mu\rho}\Gamma^{\alpha}_{\beta\nu} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\rho} , \tag{A.4}$$

which, by an index contraction, yields the Ricci tensor

$$R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} . \tag{A.5}$$

Appendix B

Here, we obtain the variation of the Lagrangian density (178) as a function of the variation of the gravitational coordinates $g^{\mu\nu}\sqrt{\ }$ and the gravitational velocities $\left(g^{\alpha\beta}\sqrt{\ }\right)_{,\nu}$, which lead to the derivatives of the Lagrangian density with these gravitational coordinates and gravitational velocities. For the first term of the Lagrangian (178) with the third expression (155), we obtain the expression

$$\begin{aligned} \delta\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} g^{\mu\nu}\sqrt{\ }\right) &= \Gamma^{\alpha}_{\mu\nu}\delta\left(\Gamma^{\beta}_{\alpha\beta} g^{\mu\nu}\sqrt{\ }\right) + \Gamma^{\beta}_{\alpha\beta} g^{\mu\nu}\sqrt{\ }\,\delta\Gamma^{\alpha}_{\mu\nu} \\ &= \Gamma^{\alpha}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{\ }_{,\nu}\right) + \Gamma^{\beta}_{\alpha\beta}\delta\left(\Gamma^{\alpha}_{\mu\nu} g^{\mu\nu}\sqrt{\ }\right) - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{\ }\right), \end{aligned} \tag{B.1}$$

which, with the second equation (156), is

$$\delta\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} g^{\mu\nu}\sqrt{\ }\right) = \Gamma^{\alpha}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{\ }_{,\alpha}\right) - \Gamma^{\beta}_{\alpha\beta}\delta\left(g^{\alpha\nu}\sqrt{\ }\right)_{,\nu} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{\ }\right). \tag{B.2}$$

For the second term of the Lagrangian (178),

$$\begin{aligned}\delta\left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}g^{\mu\nu}\sqrt{\ }\right)&=\left(\delta\Gamma^{\beta}_{\mu\alpha}\right)\Gamma^{\alpha}_{\nu\beta}g^{\mu\nu}\sqrt{\ }+\left(\delta\Gamma^{\alpha}_{\nu\beta}\right)\Gamma^{\beta}_{\mu\alpha}g^{\mu\nu}\sqrt{\ }+\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\delta\left(g^{\mu\nu}\sqrt{\ }\right)\\&=2\left(\delta\Gamma^{\beta}_{\mu\alpha}\right)\Gamma^{\alpha}_{\nu\beta}g^{\mu\nu}\sqrt{\ }+\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\delta\left(g^{\mu\nu}\sqrt{\ }\right)\\&=2\delta\left(\Gamma^{\beta}_{\mu\alpha}g^{\mu\nu}\sqrt{\ }\right)\Gamma^{\alpha}_{\nu\beta}-\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\delta\left(g^{\mu\nu}\sqrt{\ }\right)\end{aligned}\tag{B.3}$$

as, with the last equation (155),

$$\begin{aligned}g^{\mu\nu}{}_{,\alpha}&=-g^{\mu\rho}g^{\nu\sigma}g_{\rho\sigma,\alpha}=-g^{\mu\rho}g^{\nu\sigma}\left(\Gamma_{\rho\sigma\alpha}+\Gamma_{\sigma\rho\alpha}\right)\\&=-g^{\nu\sigma}\Gamma^{\mu}_{\sigma\alpha}-g^{\mu\rho}\Gamma^{\nu}_{\rho\alpha}\ .\end{aligned}\tag{B.4}$$

We obtain

$$\begin{aligned}\delta\left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}g^{\mu\nu}\sqrt{\ }\right)&=\delta\left(g^{\mu\nu}\Gamma^{\beta}_{\mu\alpha}\sqrt{\ }+g^{\mu\nu}\Gamma^{\beta}_{\mu\alpha}\sqrt{\ }\right)\Gamma^{\alpha}_{\nu\beta}-\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\delta\left(g^{\mu\nu}\sqrt{\ }\right)\\&=\delta\left(g^{\mu\nu}\Gamma^{\beta}_{\mu\alpha}\sqrt{\ }+g^{\mu\beta}\Gamma^{\nu}_{\mu\alpha}\sqrt{\ }\right)\Gamma^{\alpha}_{\nu\beta}-\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\delta\left(g^{\mu\nu}\sqrt{\ }\right)\\&=-\delta\left(g^{\nu\beta}{}_{,\alpha}\sqrt{\ }\right)\Gamma^{\alpha}_{\nu\beta}-\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\delta\left(g^{\mu\nu}\sqrt{\ }\right).\end{aligned}\tag{B.5}$$

With the two terms (B.2) and (B.5), the variation of the Lagrangian density (178) is

$$\begin{aligned}\delta\mathcal{L}=&\underline{\Gamma^{\alpha}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{\ }_{,\alpha}\right)}-\Gamma^{\beta}_{\alpha\beta}\delta\left(g^{\alpha\nu}\sqrt{\ }\right)_{,\nu}-\underline{\underline{\Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{\ }\right)}}\\&+\underline{\delta\left(g^{\nu\beta}{}_{,\alpha}\sqrt{\ }\right)\Gamma^{\alpha}_{\nu\beta}}+\underline{\underline{\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\delta\left(g^{\mu\nu}\sqrt{\ }\right)}}\\=&\underline{\Gamma^{\alpha}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{\ }\right)_{,\alpha}}-\Gamma^{\beta}_{\alpha\beta}\delta\left(g^{\alpha\nu}\sqrt{\ }\right)_{,\nu}+\underline{\underline{\left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}-\Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\right)\delta\left(g^{\mu\nu}\sqrt{\ }\right)}},\end{aligned}\tag{B.6}$$

which, with some changes of indexes, takes a form

$$\begin{aligned}\delta\mathcal{L}\ &=\Gamma^{\nu}_{\alpha\beta}\delta\left(g^{\alpha\beta}\sqrt{\ }\right)_{,\nu}-\Gamma^{\beta}_{\alpha\beta}\delta\left(g^{\alpha\nu}\sqrt{\ }\right)_{,\nu}+\left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}-\Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\right)\delta\left(g^{\mu\nu}\sqrt{\ }\right)\\&=\left(\Gamma^{\nu}_{\alpha\beta}-g^{\nu}_{\beta}\Gamma^{\sigma}_{\alpha\sigma}\right)\delta\left(g^{\alpha\beta}\sqrt{\ }\right)_{,\nu}+\left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}-\Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\right)\delta\left(g^{\mu\nu}\sqrt{\ }\right)\\&=\frac{\partial\mathcal{L}}{\partial\left(g^{\alpha\beta}\sqrt{\ }\right)_{,\nu}}\delta\left(g^{\alpha\beta}\sqrt{\ }\right)_{,\nu}+\frac{\partial\mathcal{L}}{\partial\left(g^{\mu\nu}\sqrt{\ }\right)}\delta\left(g^{\mu\nu}\sqrt{\ }\right),\end{aligned}\tag{B.7}$$

which leads to equations (165) and (166).

Appendix C

We calculate the Lagrangian density (178) of a gravitational wave in the first order approximation of the metric tensor. With (175) and (176), and the normalization condition (105), the two terms of this Lagrangian density are:

$$\begin{aligned}\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}g^{\mu\nu}\sqrt{\ }&=\frac{1}{2}\left(g^{\alpha}_{\mu}l_{\nu}+g^{\alpha}_{\nu}l_{\mu}-g_{\mu\nu}l^{\alpha}\right)u2ul_{\alpha}g^{\mu\nu}\sqrt{\ }\\&=\left(g^{\alpha\nu}l_{\nu}+g^{\alpha\mu}l_{\mu}-4l^{\alpha}\right)u^{2}l_{\alpha}\sqrt{\ }=-2u^{2}l^{\alpha}l_{\alpha}\sqrt{\ }=0,\end{aligned}\tag{C.1}$$

$$-\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu}g^{\mu\nu}\sqrt{\ } = -\frac{1}{4}\left(g^{\alpha}_{\beta}l_{\mu}+g^{\alpha}_{\mu}l_{\beta}-g_{\mu\beta}l^{\alpha}\right)\left(g^{\beta}_{\alpha}l_{\nu}+g^{\beta}_{\nu}l_{\alpha}-g_{\alpha\nu}l^{\beta}\right)g^{\mu\nu}\sqrt{\ }$$

$$= \underbrace{-\frac{1}{4}\left(g^{\alpha}_{\beta}l_{\mu}+g^{\alpha}_{\mu}l_{\beta}\right)\left(g^{\beta}_{\alpha}l_{\nu}+g^{\beta}_{\nu}l_{\alpha}\right)g^{\mu\nu}\sqrt{\ }}_{A} \underbrace{-\frac{1}{4}g_{\mu\beta}l^{\alpha}g_{\alpha\nu}l^{\beta}g^{\mu\nu}\sqrt{\ }}_{B} \tag{C.2}$$

$$\underbrace{+\frac{1}{4}\left(g^{\alpha}_{\beta}l_{\mu}+g^{\alpha}_{\mu}l_{\beta}\right)g_{\alpha\nu}l^{\beta}g^{\mu\nu}\sqrt{\ }}_{C} \underbrace{+\frac{1}{4}g_{\mu\beta}l^{\alpha}\left(g^{\beta}_{\alpha}l_{\nu}+g^{\beta}_{\nu}l_{\alpha}\right)g^{\mu\nu}\sqrt{\ }}_{D}.$$

where

$$A = -\frac{1}{4}\left(g^{\alpha}_{\beta}l_{\mu}g^{\beta}_{\alpha}l_{\nu}+g^{\alpha}_{\mu}l_{\beta}g^{\beta}_{\nu}l_{\alpha}+g^{\alpha}_{\beta}l_{\mu}g^{\beta}_{\nu}l_{\alpha}+g^{\alpha}_{\mu}l_{\beta}g^{\beta}_{\alpha}l_{\nu}\right)g^{\mu\nu}\sqrt{\ }$$

$$= -\frac{1}{4}\left(4l_{\mu}l^{\mu}+l_{\mu}l_{\nu}g^{\mu\nu}+g^{\alpha}_{\nu}l_{\alpha}l_{\mu}g^{\mu\nu}+g^{\beta}_{\mu}l_{\beta}l_{\nu}g^{\mu\nu}\right)\sqrt{\ }=0, \tag{C.3}$$

$$B = -\frac{1}{4}\underline{g_{\mu\beta}}\underline{\underline{l^{\alpha}g_{\alpha\nu}}}l^{\beta}\underline{g^{\mu\nu}}\sqrt{\ } = -\frac{1}{4}g^{\nu}_{\beta}l_{\nu}l^{\beta}\sqrt{\ }=0, \tag{C.4}$$

$$C = \frac{1}{4}\left(g^{\alpha}_{\beta}l_{\mu}+g^{\alpha}_{\mu}l_{\beta}\right)g_{\alpha\nu}l^{\beta}g^{\mu\nu}\sqrt{\ } = \frac{1}{4}\left(g^{\alpha}_{\beta}l_{\mu}+g^{\alpha}_{\mu}l_{\beta}\right)g^{\mu}_{\alpha}l^{\beta}\sqrt{\ }$$

$$= \frac{1}{4}\left(g^{\mu}_{\beta}l_{\mu}+4l_{\beta}\right)l^{\beta}\sqrt{\ }=0, \tag{C.5}$$

$$D = \frac{1}{4}\underline{g_{\mu\beta}}l^{\alpha}\left(g^{\beta}_{\alpha}l_{\nu}+g^{\beta}_{\nu}l_{\alpha}\right)\underline{g^{\mu\nu}}\sqrt{\ } = \frac{1}{4}g^{\nu}_{\beta}\left(l^{\alpha}g^{\beta}_{\alpha}l_{\nu}+l^{\alpha}g^{\beta}_{\nu}l_{\alpha}\right)=0. \tag{C.6}$$

With these expressions, we obtain a null Lagrangian density,

$$\mathcal{L}=0. \tag{C.7}$$